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Super W_3 Ward identities and differential equations of correlation functions on a supertorus

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Abstract. Using the operator formalism, Ward identities are derived for super W_3 conformal field theories on a supertorus. Differential equations of correlation functions on the supertorus are also derived by using the Ward identities and null vectors.

1. Introduction

The study of extended conformal and superconformal field theories has been playing an increasingly important role in the recent development of conformal and superconformal field theories. Among them the W theories have attracted much attention [1–21] since the original paper [1] by Zamolodchikov appeared. Fateev and Zamolodchikov [2] investigated in detail the W_3 algebra and constructed the degenerate representations of the algebra by a Feigin–Fuchs construction. A spin-N generalization (W_N algebra) has been given [3] and the explicit expressions of the W_4 [4] and W_6 [5] have also been given. The super W_3 algebra was examined in [6–8] and the $N = 2$ super W algebras have also been examined [9–11]. All of these extended conformal and superconformal field theories are formulated over Riemann surfaces of genus $g = 0$. As emphasized by Polyakov [22], Cardy [23] and Bernard [24], we have to examine these theories on a torus and higher genus Riemann surfaces in order to find a complete description of all conformal and super conformal field theories. Recently, considerable progress has been made in the study of the extended conformal field theories on higher genus Riemann surfaces [25].

When one considers an extended conformal field theory on a higher genus Riemann surface, the theory acquires complex analytical structures and the combined use of algebraic and analytic methods provides further information on the properties of the theory. The correlation functions of the theory depend holomorphically on the modular parameters and local coordinates of the Riemann surface. The extended conformal symmetry of the theory must lead to the existence of the Riemann surface. As is well known, ward identities in a theory play an important role in determining correlation functions of the theory. For example, by using the conformal and current Ward identities on a torus, one can derive the differential equations for characters of Virasoro and Kac–Moody algebras [24, 26, 27]. A general formulation for deriving Ward identities on general Riemann surfaces was given by Eguchi and Ooguri [28]. The Ward identities of the $N = 1$ superconformal algebra on general super Riemann surfaces have also been given [29, 30]. In a previous letter [31], we derived the Ward identities for the W_3 algebra on a torus.

In this paper we derive the Ward identities for the W_3 algebra on a supertorus, which are useful in calculating the correlation functions on the supertorus. Because the case of odd spin structures is more complicated than that of even spin structures we will give the derivation of Ward identities only for the former and results for the latter. We also derive differential equations satisfied by correlation functions on the supertorus in the super W_3 conformal field theory.

The rest of this paper is organized as follows. In section 2 we review the definition of a supertorus and spin structures and express correlations on the supertorus in a form of trace. We derive the Ward identities of the super W_3 algebra on the supertorus in section 3. Derivation of differential equations for correlation functions is given in section 4. The last section contains the concluding remarks and discussions.

2. Correlation functions on the supertorus

The genus 1 super Riemann surface, the supertorus, can be generated by two translations on the super complex plane specified by even and odd coordinates (z, θ) [24, 30, 32]. There are four spin structures on the supertorus since there are two types of boundary conditions, periodic $(+)$ and anti-periodic $(-)$, for the odd coordinate θ and two directions of periodic boundary conditions, $z \longrightarrow z+1$ and $z \longrightarrow z+\tau$ (corresponding to a-cycle and b-cycle), for the even coordinate z. For the odd spin structure $(+,+)$, the generators of the translation are

$$z' = z + 1 \qquad \theta' = \theta \tag{1a}$$

and

$$z' = z + \tau + \theta\delta \qquad \theta' = \theta + \delta \tag{1b}$$

where τ and δ are moduli and supermoduli parameters respectively.

For the even spin structures with $(a_1, a_2) = (+,-)$, $(-,-)$ and $(-,+)$, they can be taken to be

$$z' = z + 1 \qquad \theta' = a_1\theta \tag{2a}$$

and

$$z' = z + \tau \qquad \theta' = a_2\theta \,. \tag{2b}$$

For conformal field theories, in the operator formalism the correlation functions on the torus can be expressed as

$$\langle \phi_1(u_1) \ldots \phi_n(u_n) \rangle = \mathrm{tr}(q^{L_0 - c/24} \phi_1(u_1) \ldots \phi_n(u_n))/Z(\tau) \,. \tag{3}$$

where $q = \exp(2\pi i\tau)$ and $Z(\tau) = \mathrm{tr}(q^{L_0 - c/24})$. The trace should be taken to realize the periodicity in the b-cycle (the τ direction) while the periodicity in the a-cycle is reflected by using a cylinder coordinate $z = \exp(-2\pi i u)$.

For $N = 1$ superconformal field theories spin structure should be taken into account and one has [30, 33]

$$\langle X\rangle \equiv \langle \phi_1(u_1,\xi_1)\dots\phi_n(u_n,\xi_n)\rangle = \mathrm{tr}((-1)^F q^{L_0-c/24}\exp(-(-2\pi \mathrm{i})^{1/2}\delta G_0)\phi_1(u_1,\xi_1)\dots\phi_n(u_n,\xi_n)) \quad (4)$$

for the odd spin structure where the trace is taken in the R-sector, and

$$\langle X\rangle = \mathrm{tr}_{\mathrm{NS}}((-1)^F q^{L_0-c/24} X) \quad (5)$$

for even spin structure $(-,+)$. Similarly for $(-,-)$ we get rid of $(-)^F$ while for $(+,-)$ we get rid of $(-)^F$ and take the trace in the R-sector. In the above equations F is the fermion number. Note that we have defined the correlation functions for superconformal field theories without dividing by the partition function Z since it may be a nilpotent even Grassman number.

3. Super W_3 Ward identities on the supertorus

The generators of the super W_3 (it is also called super $W_{5/2}$; see, for example, [7, 9]) symmetry are the energy–momentum superfield $I(z,\theta) = \frac{1}{2}G(z) + \theta T(z)$ of weight $\frac{3}{2}$ and the superprimary field $J(z,\theta) = V(z) + \theta W(z)$ of weight $\frac{5}{2}$. The mode expansions of their component fields on a plane are

$$T(z) = \sum_{n\in Z} L_n z^{-n-2} \qquad G(z) = \sum_r G_r z^{-r-3/2} \quad (6)$$

$$W(z) = \sum_{n\in Z} W_n z^{-n-3} \qquad V(z) = \sum_r V_r z^{-r-5/2} \quad (7)$$

where r runs over the half-integers or integers in the NS or R sectors respectively. The mode L_n, G_r, W_n and V_r generate the super W_3 algebra as follows [8]:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$
$$[L_n, G_r] = (\tfrac{1}{2}n - r)G_{n+r}$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(R^2 - \tfrac{1}{4})\delta_{r+s,0}$$
$$[L_n, W_m] = (2n-m)W_{n+m}$$
$$[G_r, W_n] = (4r-n)V_{r+n}$$
$$[L_n, V_r] = \left(\frac{3n}{2} - r\right)V_{n+r}$$
$$\{G_r, V_s\} = W_{r+s}$$
$$[W_n, W_m] = \frac{c}{3.5!}(n^2-4)(n^2-1)n\delta_{n+m,0} + (n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m} + b^2(n-m)\Lambda_{n+m}$$

$$[W_n, V_r] = -\tfrac{1}{480}(16r^3 - 4n^3 - 12r^2 n + 8n^2 - 36r + 19n)G_{n+r} + \tfrac{7}{17}((n+2)P_{n+r} + \tfrac{2}{3}K_{n+r})$$

$$\{V_r, V_s\} = \frac{c}{360}\left(r^2 - \frac{1}{4}\right)\left(r^2 - \frac{9}{4}\right)\delta_{r+s,0} + \frac{1}{120}(6r^2 + 6s^2 - 8rs - 9)L_{r+s} + \frac{21}{136}\Lambda_{r+s} + \frac{7}{34}H_{r+s} \quad (8)$$

where the central charge c is equal to $\frac{10}{7}$ for the unitary super W_3 model, the fields P and K are descendents of the supersymmetry generator G:

$$P(z) =:\ TG\ :(Z) - \tfrac{3}{8}\partial^2 G(z) \quad (9)$$

$$K(z) = 2:\ T\partial\ :(z) - \tfrac{11}{20}\partial^3 G(z) \quad (10)$$

H is the bosonic field of weight 4 that can be constructed out of T and G such that

$$H(z) =:\ G\partial G\ :(Z) - \frac{7}{10}\partial^2 T(z) + \frac{17}{5c+22}\Lambda(z) \quad (11)$$

and

$$\Lambda(z) =:\ TT\ :(z) - \tfrac{3}{10}\partial^2 T(z)\,. \quad (12)$$

As usual, we have associated any field $\phi(z)$ of weight h with its modes ϕ_n according to

$$\phi_n(z) = \sum_n \phi_n z^{-n-h}\,. \quad (13)$$

From superconformal symmetry one can easily find that a primary superfield $\phi_i(z,\theta) = \varphi_i(z) + \theta\psi_i(z)$ of weight Δ_i in the super W_3 field theory obeys the following operator product expansion:

$$J(z_1,\theta_1)\phi_i(z_2,\theta_2) = \frac{v_i\theta_{12}\phi_i(z_2,\theta_2)}{z_{12}^3} + \frac{W_{i1}\theta_{12}\partial_{z_2}\phi_i(z_2,\theta_2)}{z_{12}^2} + \frac{1}{z_{12}}[(W'_{i1} - W_{i1})\partial_{z_2}D_2\phi_i(z_2,\theta_2) + W_{i2}\theta_{12}(\partial^2_{z_2}\phi_i(z_2,\theta_2) + W_{i3}\theta_{12}S_i(z_2,\theta_2)] + \text{reg} \quad (14)$$

where $\theta_{12} = \theta_1 - \theta_2$, $z_{12} = z_1 - z_2 - \theta_1\theta_2$, $D_2 = \partial_{\theta_2} + \theta_2\partial_{z_2}$

$$W_{i1} = \frac{3v_i}{2\Delta_i} \qquad W_{i2} = \frac{3v_i}{\Delta_i(2\Delta_i + 1)}$$
$$W_{i3} = \frac{24(\Delta_i - 1)v_i}{c(2\Delta_i + 1) + 2\Delta_i(8\Delta_i - 5)} \qquad W'_{i1} = \frac{3v_i}{2(\Delta_i + 1/2)} \quad (15)$$

and

$$S_i(z,\theta) =:\ T\phi_i\ :(z,\theta) - \frac{3}{2(2\Delta_i+1)}\partial_z^2\phi_i(z,\theta)$$

are the quasisuperprimary fields associated with the superprimary field ϕ_i. Therefore, in addition to the well known (anti-) commutation relations

$$[L_n, \phi_i(z,\theta)] = z^n\left[(n+1)(\Delta_i + \tfrac{1}{2}\theta D) + z\partial_z\right]\phi_i(z,\theta) \tag{16a}$$

$$[G_r, \phi_i(z,\theta)]_\pm = z^{r-1/2}[-\Delta_i(2r+1)\theta + z(\partial_\theta - \theta\partial_z)]\phi_i(z,\theta) \tag{16b}$$

one has

$$[W_n, \phi_i(z,\theta)] = z^n\bigg(\frac{(n+2)(n+1)}{2}v_i\phi_i(z,\theta) + zW_{i1}(n+2)\partial\phi_i + z(n+2)(W'_{i1} - W_{i1})\theta\partial D\phi_i + W_{i2}z^2\partial^2\phi + W_{i3}z^2S_i(z,\theta)\bigg) \tag{17a}$$

$$[V_n, \phi_i(z,\theta)]_\pm = -z^{n-1/2}\bigg(\theta\frac{(n+3/2)(n+1/2)}{2}v_i\phi_i(z,\theta) + W_{i1}(n+3/2)z\theta\partial\phi_i - (W'_{i1} - W_{i1})z^2\partial D\phi_i + W_{i2}\theta z^2\partial^2\phi_i + W_{i3}\theta z^2 S_i(z,\theta)\bigg) \tag{17b}$$

for the primary superfields ϕ_i of the super W_3 algebra.

We now derive the super W_3 Ward identities on the supertorus. For this purpose and for the sake of convenience, we define

$$J_n^F(u) = (-2\pi\mathrm{i})^{5/2}V_n\,\mathrm{e}^{2\pi\mathrm{i}nu} \qquad J^F(u) = \sum_n J_n^F(u)$$
$$J_n^B(u) = (-2\pi\mathrm{i})^3 W_n\,\mathrm{e}^{2\pi\mathrm{i}nu} \qquad J^B(u) = \sum_n J_n^B(u) \tag{18}$$

with $J(u,\xi) = J^F(u) + \xi J^B(u) = \sum_n J_n(u,\xi)$ being the generator of weight $\frac{5}{2}$ in the cylinder coordinate $z = \exp(-2\pi\mathrm{i}u)$, and similarly,

$$I_n^F(u) = \frac{(-2\pi\mathrm{i})^{3/2}}{2}G_n\,\mathrm{e}^{2\pi\mathrm{i}nu} \qquad I^F(u) = \sum_n I_n^F(u)$$
$$I_n^B(u) = (-2\pi\mathrm{i})^2 L_n\,\mathrm{e}^{2\pi\mathrm{i}nu} \qquad I^B(u) = \sum_n I_n^B(u) \tag{19}$$

for the energy–momentum superfield $I(u,\xi) = I^F(u) + \xi I^B(u)$.

According to (4), for the odd-spin structure, the correlation functions with an insertion of the generator $J(u_0,\xi_0)$ are given by

$$\begin{aligned}\langle J(u_0,\xi_0)X\rangle &= \mathrm{tr}((-1)^F q^{L_0}\exp(-(-2\pi\mathrm{i})^{1/2}\delta G_0)J(u_0,\xi_0)X) \\ &= \sum_n \mathrm{tr}((-1)^F q^{L_0}\exp(-(-2\pi\mathrm{i})^{1/2}\delta G_0)J_n(u_0,\xi_0)X)\end{aligned} \tag{20}$$

where we have omitted the constant factor $q^{c/24}$ for simplicity, which is irrelevant in the manipulations below.

Using a permutation invariance trace and (8), it is easy to get

$$A_n \equiv \mathrm{tr}((-1)^F q^{L_0} \exp(-(-2\pi \mathrm{i})^{1/2}\delta G_0) J_n(u_0, \xi_0) X)$$
$$= \frac{1}{1-Q^n}[(J_n^F(u_0) + (\tilde{\delta} + \xi_0) J_n^B(u_0)), X] \tag{21}$$

for $n \neq 0$, where $Q^n = \exp(2\pi \mathrm{i} n(\tau + \xi_0\delta))$ and $\tilde{\delta} = [Q^n/(1-Q^n)]\delta$. Calculating the commutator in (21) by using (17) and substituting (21) into (20), one obtains the following Ward identity for one insertion of the generator $J(u, \xi)$ [35]:

$$\begin{aligned}
\langle J(u_0, \xi_0) X\rangle = \langle J_0(u_0, \xi_0) X\rangle \\
&+ \sum_{i=1}^{N}\Big\{ \frac{v_i}{2}\Big[\frac{\mathrm{i}\delta}{4\pi} H(u_{0i}|\tau_{0i}) + \frac{\partial^2}{\partial^2_{u_0}}(\xi_0 - \xi_i)(\mathcal{P}(u_{0i}|\tau_{0i}) + 2\eta_1(\tau_{0i}))\Big] \\
&+ W_{i1}\Big[(\xi_0 - \xi_i)\partial_{u_0}(\zeta(u_{0i}|\tau_{0i}) - 2\eta_1(\tau_{0i})u_{0i}) - \frac{\mathrm{i}\delta}{4\pi}\partial_{u_0} H(u_{0i}|\tau_{0i})\Big]\partial_{u_i} \\
&- (W'_{i1} - W_{i1})(\zeta(u_{0i}|\tau_{0i}) - 2\eta_1(\tau_{0i})u_{0i})\partial D_i \\
&+ \tilde{W}_{i2}\Big[\frac{i\delta}{4\pi} H(u_{0i}|\tau_{0i}) - (\xi_0 - \xi_i)(\zeta(u_{0i}|\tau_{0i}) - 2\eta_1(\tau_{0i})u_{0i})\Big]\partial^2_{u_i} \\
&- (\xi_0 - \xi_i) W_{i3}(\zeta(u_{0i}|\tau_{0i}) - 2\eta_1(\tau_{0i})u_{0i})\tilde{L}^i_{-2}\Big\}\langle X\rangle
\end{aligned} \tag{22}$$

where

$$u_{0i} = u_0 - u_i - \xi_0\xi_i \qquad \tau_{0i} = \tau + (\xi_0 + \xi_i)\delta \qquad D_i = \partial_{\xi_i} + \xi_i\partial_{u_i}$$

$$\tilde{W}_{i2} = W_{i2} - \frac{3W_{i3}}{2(2\Delta_i + 1)} \qquad H(u|\tau) = 8\pi^2 \sum_{n\neq 0} \frac{q^n}{(1-q^n)^2}\exp(2\pi \mathrm{i} n u)$$

$$\mathcal{P}(u|\tau) = (-2\pi \mathrm{i})^2 \sum_{n\neq 0} \frac{n}{1-q^n}\exp(2\pi \mathrm{i} n u) - 2\eta_1(\tau)$$

$$\zeta(u|\tau) = (-2\pi \mathrm{i}) \sum_{n\neq 0} \frac{1}{1-q^n}\exp(2\pi \mathrm{i} n u) + 2\eta_1(\tau)u\,.$$

and $\eta_1(\tau) = 2\pi \mathrm{i}\partial_\tau \ln \eta(\tau)$ with $\eta(\tau)$ being the Dedekind η-function. In (22), the operator $\tilde{L}^i_{-2}$ is defined by

$$\begin{aligned}
\tilde{L}^i_{-2}\langle X\rangle = \Big\{ 2\pi \mathrm{i}\partial_\tau + \sum_{i=1}^{N}\{\Delta_i[(-2\pi \mathrm{i})^2\eta_3 + (-2\pi \mathrm{i})^2\delta\xi\eta_4)] + \tfrac{1}{2}((-2\pi \mathrm{i})^2(\xi_i\eta_3 - \eta_2\delta))D_i \\
+ (\zeta(0|\tau) - (-2\pi \mathrm{i})^2\eta_3\delta\xi_i)\partial_{u_i}\}\Big\}\langle X\rangle\,.
\end{aligned} \tag{23}$$

In a similar way, by means of (16), (17), (21) and the super W_3 algebra (8), one derives the Ward identities for two insertions of the generators:

$$
\begin{aligned}
\langle J(u_0,\xi_0)I(u_1,\xi_1)X\rangle &= \langle J_0(u_0,\xi_0)IX\rangle + \Big\{\frac{5}{2}\Big[(\xi_0-\xi_1)(\mathcal{P}(u_{01}|\tau_{01})+2\eta_1(\tau_{01}))\\
&+\frac{\mathrm{i}\delta}{4\pi}\partial_{u_0}H(u_{01}|\tau_{01})\Big] + \frac{1}{2}\zeta(u_{01}|\tau_{01})D_1 + \Big[(\xi_0-\xi_1)\zeta(u_{01}|\tau_{01})\\
&-\frac{\mathrm{i}\delta}{4\pi}H(u_{01}|\tau_{01})\Big]\partial_{u_1}\Big\}\langle J(u_1,\xi_1)X\rangle\\
&+\sum_{i=2}^{N+2}\Big\{\frac{v_i}{2}\Big[\frac{\mathrm{i}\delta}{4\pi}\partial^2_{u_i}H(u_{0i}|\tau_{0i})+\partial_{u_0}(\xi_0-\xi_i)\mathcal{P}(u_{0i}|\tau_{0i})+2\eta_1(\tau_{0i})\Big]\\
&+W_{1i}\Big[(\xi_0-\xi_i)\partial_0(\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i})u_{0i})-\frac{\mathrm{i}\delta}{4\pi}\partial_0 H(u_{0i}|\tau_{0i})\Big]\partial_i\\
&-(W'_{1i}-W_{1i})(\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i})u_{0i})\partial D_i\\
&+\tilde{W}_{2i}\Big[\frac{-\mathrm{i}\delta}{4\pi}H(u_{0i}|\tau_{0i})-(\xi_0-\xi_i)(\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i})u_{0i})\Big]\partial_i^2\\
&-W_{3i}(\xi_0-\xi_i)((\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i})u_{0i})\tilde{L}^i_{-2}\Big\}\langle I(u_1,\xi_1)X\rangle
\end{aligned}\tag{24}
$$

and

$$
\begin{aligned}
\langle J(u_0,\xi_0)J(u_1,\xi_1)X\rangle &= \langle J_0(u_0,\xi_0)JX\rangle - \frac{c}{3.5!}\partial_0^4(\zeta(u_{01}|\tau_{01})\\
&-2\eta_1(\tau_{01})u_{01})\langle I(u_1,\xi_1)X\rangle + \frac{1}{3!}[(\xi_0-\xi_1)\partial_0^2(\zeta(u_{01}|\tau_{01})-2\eta_1(\tau_{01})u_{01}]\\
&-\frac{\mathrm{i}\delta}{4\pi}\partial_0^3H(u_{01}|\tau_{01}))\langle IX\rangle - \frac{1}{3}(\partial_0^2(\zeta(u_{01}|\tau_{01})-2\eta_1(\tau_{01})u_{01}))D_1\langle IX\rangle\\
&+\frac{1}{3}\Big\{\frac{\mathrm{i}\delta}{4\pi}\partial_0^2H(u_{01}|\tau_{01})-(\xi_0-\xi_1)\partial_0^2(\zeta(u_{01}|\tau_{01})-2\eta_1(\tau_{01})u_{01})\Big\}\partial_1\langle IX\rangle\\
&+\frac{1}{6}\partial_0(\zeta(u_{01}|\tau_{01})-2\eta_1(\tau_{01})u_{01})\partial D_1\langle IX\rangle\\
&+\Big\{\frac{\mathrm{i}\delta}{4\pi}\partial_0H(u_{01}|\tau_{01})-(\xi_0-\xi_1)(\mathcal{P}(u_{01}|\tau_{01})+2\eta_1(\tau_{01})\Big\}\\
&\times\Big[\frac{1}{4}\partial_1^2\langle IX\rangle+\frac{7}{17}\langle\Gamma_0X\rangle+\frac{337\xi_1}{408}\langle\Lambda X\rangle\Big]-(\zeta(u_{01}|\tau_{01})-2\eta_1(\tau_{01})u_{01})\\
&\times\Big(\frac{1}{20}\partial^2D_1\langle IX\rangle-\frac{7}{17}D_1\langle\Gamma_0X\rangle+\frac{21}{136}\langle\Lambda X\rangle+\frac{28\xi_1}{51}\langle KX\rangle\Big)\\
&+\Big[\frac{\mathrm{i}\delta}{4\pi}H(u_{01}|\tau_{01})-(\xi_0-\xi_1)(\zeta(u_{01}|\tau_{01})-2\eta_1(\tau_{01})u_{01})\Big]\\
&\times\Big(\frac{1}{15}\partial_1^3\langle IX\rangle+\frac{14}{51}\langle KX\rangle+\frac{28}{51}D_1\langle\Lambda X\rangle\Big)
\end{aligned}
$$

$$+\sum_{i=2}^{N+2}\left\{\frac{v_i}{2}\left(\frac{i\delta}{4\pi}\partial_0^2 H(u_{0i}|\tau_{0i})+(\xi_0-\xi_i)\partial_0^2(\mathcal{P}(u_{0i}|\tau_{0i})+2\eta_1(\tau_{0i}))\right)\langle JX\rangle\right.$$

$$+W_{1i}\left((\xi_0-\xi_i)\partial_0(\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i})u_{0i})-\frac{i\delta}{4\pi}\partial_0 H(u_{0i}|\tau_{0i})\right)\partial_i\langle JX\rangle$$

$$-(W'_{1i}-W_{1i})(\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i}|u_{0i})\partial D_i\langle JX\rangle$$

$$+\tilde{W}_{2i}\left[\frac{i\delta}{4\pi}H(u_{0i}|\tau_{0i})-(\xi_0-\xi_i)(\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i})u_{0i})\right]\partial_i^2\langle JX\rangle$$

$$-W_{3i}(\xi_0-\xi_i)(\zeta(u_{0i}|\tau_{0i})-2\eta_1(\tau_{0i})u_{0i})\bar{L}^i_{-2}\langle JX\rangle \tag{25}$$

where Γ_0 is a spin-$\frac{7}{2}$ field which is defined by

$$\Gamma_0(u,\xi)=\Omega(u)-\tfrac{1}{2}\xi H(u)=\Gamma_0^F-\tfrac{1}{2}\xi\Gamma_0^B \tag{26}$$

with

$$\Omega(u)=\Gamma_0^F(u)=2(I^BI^F)(u)-\frac{3}{4}\partial^2 I^F(u)$$

and

$$\Gamma_0^B(u)=4(I^F\partial I^F)(u)-\frac{7}{10}\partial^2 I^B(u)+\frac{17}{5c+22}\Lambda(u)\,.$$

Here, $(I^BI^F)(u)$ etc denote the normal products of two field operators which can be expressed as follows:

$$(I^BI^F)(u)=\oint_{c_u}du_0\frac{1}{u_0-u}I^B(u_0)I^F(u)\,.$$

Through some tedious calculations, we obtain the following expressions of the correlation functions involving Γ_0 or Λ or K:

$$\langle\Gamma_0^F(u_0)\phi_1\dots\phi_N\rangle=\langle\Gamma_0^FX\rangle=2\left\{2\pi i\partial_\tau A+\sum_{i=1}^{N}\Delta_i\left(\mathcal{P}(u_0-u_i|\tau)+\frac{3}{2}(-2\pi i)^2\eta_3\right)A\right.$$

$$+\sum_{i=1}^{N}\left(\Delta_i\frac{i\delta\xi_i}{4\pi}\partial_0^2 H(u_0-u_i|\tau)A-\sum_{i=1}^{N}\partial_0\zeta(u_0-u_i|\tau)\xi_i D_i A\right.$$

$$-(-2\pi i)^2\frac{1}{2}\eta_2\delta B-\frac{i\delta}{8\pi}\sum_{i=1}^{N}\partial_0 D_i A+\zeta(0|\tau)\partial_0+\sum_{i=1}^{N}\zeta(u_0-u_i|\tau)A$$

$$+\sum_{i=1}^{N}\frac{i\xi_i\delta}{4\pi}\partial_0 H(u_0-u_i|\tau)A-\frac{i\pi c}{6}\sum_{i=1}^{N}\xi_i\partial_0\zeta(u_0-u_i|\tau)\langle X\rangle$$

$$\left.-\frac{i\delta c}{48\pi}\sum_{i=1}^{N}\partial_0 H(u_0-u_i|\tau)\langle X\rangle-\frac{3}{8}\partial_0^2 A\right\} \tag{27}$$

$$\langle\Gamma_0^B X\rangle = 4\Big\{\pi\mathrm{i}(\partial_\delta-\delta\partial_\tau)A+\sum_{i=1}^{N}\Big[\xi_i(\Delta_i\partial_0(\mathcal{P}(u_0-u_i|\tau)+2\eta_1(\tau))$$
$$+\Delta_i(\mathcal{P}(u_0-u_i|\tau)+2\eta_1(\tau))\partial_0)A+\frac{\mathrm{i}\delta}{4\pi}(\partial_0^2 H(u_0-u_i|\tau)$$
$$+\partial_0 H(u_0-u_i|\tau)\partial_0)A+\Big(\frac{1}{2}\xi_i\delta\partial_0^2\Big(\frac{\mathrm{i}}{2\pi}(\xi(u_0-u_i|\tau)-2\eta_1(\tau)(u_0-u_i))$$
$$+\frac{1}{8\pi^2}H(u_0-u_i|\tau)-\xi_i\delta(\mathcal{P}(u_0-u_i|\tau)+2\eta_1(\tau)\Big)\partial_0\Big)D_iA$$
$$-\xi_i(\partial_0(\xi(u_0-u_i|\tau)-2\eta_1(\tau)(u_0-u_i))+(\xi(u_0-u_i|\tau)$$
$$-2\eta_1(\tau)(u_0-u_i))\partial_0)\partial_iA-\frac{\mathrm{i}\delta}{4\pi}(H(u_0-u_i|\tau)\partial_0\partial_i+\partial_0 H(u_0-u_i|\tau)\partial_i)A\Big]$$
$$+\frac{\mathrm{i}\delta}{2\pi}(-(-2\pi\mathrm{i})^2 8\pi^2\eta_5+(2\pi\mathrm{i})16\pi^2\eta_1\partial_0)A-\frac{\mathrm{i}\delta}{4\pi}((-2\pi\mathrm{i})^2 8\pi^2\eta_2$$
$$+H(0|\tau)\partial_0)\partial_0A+\frac{c}{12}(-2\pi\mathrm{i})^4\eta_6\langle X\rangle+\frac{17}{5c+22}\langle\Lambda X\rangle-\frac{7}{10}\partial_0^2B$$

$$\langle\Lambda(u_0)X\rangle=(-2\pi\mathrm{i})\partial_\tau B+\sum_{i=1}^{N}\frac{\mathrm{i}\Delta_i\xi_i\delta}{4\pi}\partial_{u_0}^2H(u_0-u_i|\tau)B-12\mathrm{i}\pi^3\delta\eta_4A$$
$$+\frac{1}{2}\sum_{i=1}^{N}\Big(\frac{-\mathrm{i}\delta}{4\pi}\partial_{u_0}H(u_0-u_i|\tau)-(\mathcal{P}(u_0-u_i|\tau)+2\eta_1)\xi_i\Big)D_iB$$
$$-2\pi^2\delta\eta_2A+-\frac{(-2\pi\mathrm{i})^2}{2}\eta_3B-\sum_{i=1}^{N}[(-)\zeta(u_0-u_i|\tau)\partial_{u_i}+\zeta(0|\tau)\partial_0]B$$
$$-\Big(2\pi^2\delta\eta_2\partial_0-\frac{\mathrm{i}\delta}{4\pi}\sum_{i=1}^{N}\partial_0H(u_0-u_i|\tau)\partial_i\Big)A+\frac{c}{12}(-2\pi\mathrm{i})^3\eta_6\langle X\rangle$$
$$-\frac{3}{10}\partial_0^2B \tag{28}$$

and

$$\langle K(u_0,\xi_0)X\rangle=4\Big\{2\pi\mathrm{i}\partial_\tau\partial_0A+\sum_{i=1}^{N}\Big[\Delta_i(\partial_0(\mathcal{P}(u_0-u_i|\tau)+2\eta_1(\tau))+(\mathcal{P}(u_0-u_i|\tau)$$
$$+2\eta_1(\tau))\partial_0+\frac{3}{2}(-2\pi\mathrm{i})^2\eta_3\partial_0)A+\Delta_i\frac{\mathrm{i}\delta\xi_i}{4\pi}(\partial_0^3H(u_0-u_i|\tau)$$
$$+\partial_0^2H(u_0-u_i|\tau)\partial_0)A-(\partial_0^2(\xi(u_0-u_i|\tau)-2\eta_1(\tau)(u_0-u_i))\xi_iD_i$$
$$+\partial_0(\xi(u_0-u_i|\tau)-2\eta_1(\tau)(u_0-u_i))\xi_iD_i\partial_0)A+\tfrac{1}{2}(-2\pi\mathrm{i})^2\delta\eta_2\partial_0B$$
$$-\frac{\mathrm{i}\delta}{8\pi}(\partial_0^2H(u_0-u_i|\tau)D_i+\partial_0H(u_0-u_i|\tau)D_i\partial_0)A+(\zeta(0|\tau)\partial_0$$
$$+(\xi(u_0-u_i|\tau)-2\eta_1(\tau)(u_0-u_i))\partial_i)\partial_0A+\partial_0(\xi(u_0-u_i|\tau)$$

$$-2\eta_1(\tau)(u_0-u_i))\partial_i A+\frac{\mathrm{i}\xi_i\delta}{4\pi}(\partial_0^2 H(u_0-u_i|\tau)+\partial_0 H(u_0-u_i|\tau)\partial_0)A$$
$$-\frac{\mathrm{i}\pi c}{6}\xi_i\partial_0^3(\xi(u_0-u_i|\tau)-2\eta_1(\tau)(u_0-u_i))\langle X\rangle$$
$$-\frac{\mathrm{i}c\delta}{48\pi}\partial_0^2 H(u_0-u_i|\tau)\langle X\rangle\Big]\Big\}-\frac{11}{10}\partial_0^3 A \tag{29}$$

where

$$B=\Big\{2\pi\mathrm{i}\partial_\tau+\sum_{i=1}^{N}\Big\{\Delta_i\Big[\mathcal{P}(u_0-u_i|\tau)+\frac{\mathrm{i}\xi_i\delta}{4\pi}\partial_0^2 H(u_0-u_i|\tau)\Big]$$
$$+\Big(-\frac{1}{4}\xi_i\partial_0(\zeta(u_0-u_i|\tau)-2\eta_1(\tau)(u_0-u_i))+\frac{\mathrm{i}\delta}{8\pi}\partial_0 H(u_0-u_i|\tau)\Big)D_i$$
$$+\Big(\zeta(u_0-u_i|\tau)-\frac{\mathrm{i}\xi_i\delta}{4\pi}\partial_0 H(u_0-u_i|\tau)\partial_i\Big\}\Big\}\langle X\rangle \tag{30}$$

and

$$A=\Big\{\pi\mathrm{i}(\partial_\delta-\delta\partial_\tau)-\sum_{i=1}^{N}\Big[\xi_i(\Delta_i(\mathcal{P}(u_0-u_i|\tau)-2\eta_1)+\frac{\mathrm{i}\delta}{4\pi}\Delta_i\partial_i\partial_0 H(u_0-u_i|\tau)$$
$$+\frac{1}{2}(\zeta(u_0-u_i|\tau)\partial)+\frac{\mathrm{i}\delta\xi_i}{4\pi}\partial_0 H(u_0-u_i|\tau))D_i$$
$$-\frac{\mathrm{i}\delta}{4\pi}H(u_0-u_i|\tau)\partial_i\Big]\Big\}\langle X\rangle\,. \tag{31}$$

In the above equations, for convenience, we have defined a series of η-functions which relate to the elliptic functions as follows:

$$\hat{\eta}_1=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}=-(-2\pi\mathrm{i})^{-2}\eta_1+\frac{1}{24}\tilde{\eta}_1=\sum_{n=1/2,3/2\dots}^{\infty}\frac{2nq^n}{1-q^n}$$

$$\eta_2=\sum_{n\neq0}\frac{nq^n}{(1-q^n)^2}=(-2\pi\mathrm{i})^{-1}q\frac{\partial}{\partial q}(\zeta(0|\tau))$$

$$\eta_3=\sum_{n\neq0}\frac{n}{1-q^n}=(-2\pi\mathrm{i})^{-2}(\mathcal{P}(0|\tau)+2\eta_1)$$

$$\eta_4=\sum_{n\neq0}\frac{n^2q^n}{(1-q^n)^2}=-2(-2\pi\mathrm{i})^{-2}q\frac{\partial}{\partial q}\eta_1$$

$$\eta_5=\sum_{n\neq0}\frac{n^2q^n}{1-q^n}=(-2\pi\mathrm{i})^{-3}\frac{\partial\mathcal{P}}{\partial u}\Big|_{u=0}$$

$$\eta_6=\sum_{n\neq0}\frac{n^3}{1-q^n}=(-2\pi\mathrm{i})^{-4}\frac{\partial^2\mathcal{P}}{\partial u^2}\Big|_{u=0}\,.$$

The $N=1$ superconformal Ward identities are [29, 30]

$$\langle I(u,\xi)X\rangle = \Big\{\pi\mathrm{i}(\partial_\delta - \delta\partial_\tau)2\pi\mathrm{i}\xi_0\partial_\tau + \sum_{i=1}^{N}\Big[\Delta_i\Big((\xi_0-\xi_i)\mathcal{P}_{0i} + \frac{\mathrm{i}\delta}{4\pi}\partial_0 H_{0i}\Big) + \frac{1}{2}\zeta_{0i}D_i + \Big((\xi_0-\xi_i)\zeta_{0i} - \frac{\mathrm{i}\delta}{4\pi}H_{0i}\Big)\partial_i\Big]\Big\}\langle X\rangle \tag{32}$$

and

$$\begin{aligned}\langle I(u_0,\xi_0)I(u_1,\xi_1)\ldots I(u_m,\xi_m)\phi_{m+1}\cdots\phi_n\rangle &= \{\pi\mathrm{i}(\partial_\delta-\delta\partial_\tau)+2\pi\mathrm{i}\partial_\tau\}\langle I(u_1,\xi_1)\ldots\phi_n\rangle \\ &\quad + \sum_{j=1}^{n}\Big\{\Delta_j\Big[\xi_{0j}\mathcal{P}_{0j} + \frac{\mathrm{i}\delta}{4\pi}\partial_0 H_{0j}\Big] + \frac{1}{2}\zeta_{0j}D_j + \Big[\xi_{0j}\zeta_{0j} - \frac{\mathrm{i}\delta}{4\pi}H_{0j}\Big]\partial_j\Big\} \\ &\quad + \frac{c}{12}\sum_{j=1}^{m}(-1)^j\partial_0 P_{0j}\langle I(u_1,\xi_1)\ldots I(u_{j-1},\xi_{j-1})I(u_{j+1},\xi_{j+1}\cdots\phi_n\rangle\end{aligned} \tag{33}$$

(note that $\Delta_1 = \cdots = \Delta_m = \frac{3}{2}$), where

$$\mathcal{P}_{0i} = \mathcal{P}(u_{0i}|\tau_{0i}) + 2\eta_1(\tau_{0i}) \qquad \zeta_{0i} = \zeta(u_{0i}|\tau_{0i}) - 2\eta_1 u_{0i} \qquad H_{0i} = H(u_{0i}|\tau_{0i})\,.$$

Putting (22), (24), (25), (32) and (33) together, we have the super W_3 Ward identities on the supertorus of the odd spin structure.

In the case of the even spin structures we proceed in a similar way and the results are given in the appendix.

4. Differential equations for correlation functions

In the super W_3 conformal field theories null vectors appear if the parameters (i.e. the eignvalues of L_0 and W_0) Δ, v satisfy some algebraic equations, like the case in W_3 conformal field theories. Because the local algebraic structure remains the same on the supertorus we can still use the null vectors to derive differential equations for correlation functions on the supertorus of the W_3 superconformal field theories. In this section we give, as examples, the first few null vectors in the NS sector and derive the differential equations satisfied by correlation functions on the supertorus using the null vectors and the super W_3 Ward identities.

The null vector χ_N satisfies the conditions

$$\begin{aligned}&L_n\chi_N = G_n\chi_N = W_n\chi_N = V_n\chi_N = 0 \qquad \text{for} \quad n>0 \\ &L_0\chi_N = (\Delta+N)\chi_N\end{aligned} \tag{34}$$

with some positive half-integer or integer N. From (8) and (34), a little calculation leads to:

I. $N = \frac{1}{2}$:

$$\chi_{1/2} = \left(G_{-1/2} - \frac{2\Delta}{v} V_{-1/2}\right)\phi_{\Delta,v} \tag{35}$$

where $\phi_{\Delta,v}$ is the super W_3 primary field if

$$3v^2 = \Delta^2\left[\frac{27}{22+5c}\left(\Delta + \frac{1}{3}\right) - \frac{1}{5}\right] \tag{36}$$

and II. $N = 1$:

$$\chi_1 = \left(L_{-1} - \frac{1+2\Delta}{3v} W_{-1} + v^{-1} G_{-1/2} V_{-1/2}\right)\phi_{\Delta,v} \tag{37}$$

if

$$18v^2 = \Delta\left[\frac{1}{22+5c}(\Delta + 1/5)(128\Delta - 17) - \frac{1}{5}(4\Delta - 11)\right]. \tag{38}$$

Substituting (35) into

$$\langle \chi_{1/2}(z_0, \theta_0)\phi_1(z_1, \theta_1)\ldots\phi_N(z_N, \theta_N)\rangle = 0$$

and using the Ward identities (A1) and (A2), one obtains the following differential equation:

$$\begin{aligned}
&\sum_{i=1}^{N} 2\left(\theta_i\partial_i + \frac{1}{2}D_i\right)\langle\phi_0\ldots\phi_N\rangle - \frac{2\Delta}{v}\sum_{i=1}^{N}\Big\{\Big[\theta_i W_{i1}\partial_i + (W'_{i1} - W_{i1})(z_i - z_0)\partial_i D_i \\
&\quad + \tilde{W}_{2i}(z_i - z_0)\theta_i\partial_i^2 + W_{3i}(z_i - z_0)\theta_i\Big[\sum_{j\neq i}^{N}(F_I(z_i, z_j|\tau)\partial_j + \tfrac{1}{2}D_j F_I D_j \\
&\quad + \Delta_j\partial_j F_I) + z_i^{-2}[-2z_i\partial_i + \tfrac{1}{2}\theta_i(-1+2\hat{\eta}_1)D_i + \Delta_i(-1+2\hat{\eta}_1)]\Big]\langle\phi_0 X\rangle \\
&\quad + W_{3i}(z_i - z_0)\theta_i z_i^{-2}\frac{\partial}{Z\partial(2\pi\tau)}(Z\langle\phi_0 X\rangle)\Big\} = 0
\end{aligned} \tag{39}$$

where $\phi_0 \equiv \phi_{\Delta,v}$ and $X \equiv \phi_1\ldots\phi_N$.

Similarly, from (37) and (A1) and (A2), one derives the differential equations as follows:

$$\begin{aligned}
&\sum_{i=1}^{N}\partial_i\langle\phi_0\ldots\phi_N\rangle - \frac{1+2\delta}{3v}\sum_{i=1}^{N}\Big[\Big(W_{i1}\partial_i + (W'_{i1} - W_{i1})\theta_i\partial_i D_i + \tilde{W}_{2i}(z_i - z_0)\partial_i^2 \\
&\quad + W_{3i}(z_i - z_0)\sum_{j\neq i}^{N}(F_I(z_i, z_j|\tau)\partial_i\tfrac{1}{2}D_j F_I D_j + \Delta_j\partial_j F_I) \\
&\quad + z_i^{-2}[-2z_i\partial_i + \tfrac{1}{2}\theta_i(-1+2\hat{\eta}_1)D_i + \Delta_i(-1+2\hat{\eta}_1)]\Big)\langle\phi_0\ldots\phi_N\rangle \\
&\quad + W_{3i}(z_i - z_0)z_i^{-2}\frac{\partial}{Z\partial(2\pi\tau)}(Z\langle\phi_0 X\rangle)\Big] + \frac{1}{v}C = 0
\end{aligned} \tag{40}$$

where

$$C = \langle G_{-1/2}V_{-1/2}\phi_0 X\rangle = 2\sum_{j=1,i=1}^{N}[\theta_j\partial_j+\tfrac{1}{2}D_j][W_{1i}\partial_i\theta_i+(W'_{1i}-W_{1i})(z_i-z_0)\partial_i D_i$$

$$+\tilde{W}_{2i}(z_i-z_0)\theta_i A-\mathrm{i}]\langle\phi_0 X\rangle+\sum_{i=1} W_{3i}(z_i-z_0)z_i^{-2}\frac{\partial}{Z\partial(2\pi\tau)}(Z\langle\phi_0 X\rangle)$$

$$+\tfrac{1}{2}\sum_{i=1}^{N}[W_{1i}\partial_i+(W'_{1i}-W_{1i})\theta_i(z_i-z_0)\partial_i D_i+\tilde{W}_{2i}(z_i-z_0)\partial_i^2$$

$$+W_{3i}(z_i-z_0)A_i]\langle\phi_0 X\rangle \qquad (41)$$

with

$$A_i = \sum_{j\neq i}^{N}[F_I(z_i,z_j|\tau)\partial_j+\tfrac{1}{2}D_jF_ID_j+\Delta_j\partial_jF_I]$$

$$+z_i^{-2}[2z_i\partial_i+\tfrac{1}{2}\theta_i[-1+2\hat{\eta}_1]D_i+\Delta_i(-1+2\hat{\eta}_1)]\,. \qquad (42)$$

5. Summary and discussion

In summary, we have derived the Ward identities on a supertorus for super W_3 conformal field theories by using the operator formalism. For even spin structure, when setting the Grassmann variables equal to zero the Ward identities reduce to those in W_3 conformal field theories, as expected, since the super W_3 algebra contains the W_3 algebra as a subalgebra. We have also derived differential equations for correlation functions on the supertorus by using null vectors of level $\frac{1}{2}$ and 1 and the super W_3 Ward identities.

In the Ward identities for insertions of J (see, for examples, (22) and (25)) there still remain the correlation functions involving the zero mode J_0 on the right-hand side of the identities. In order to have a complete description of super W_3 Ward identities on the supertorus in the sense that the Ward identities entirely determine the correlation functions with any insertions of generators in terms of those without any insertion of generators inside, one has to deal with $\langle W_0\phi_1\ldots\phi_N\rangle$ which has been investigated in [31]; only $\langle V_0\phi_1\ldots\phi_N\rangle$ needs to be treated. One way is to introduce the character-valued correlation functions

$$\langle\phi_1\ldots\phi_n\rangle_\beta \equiv \mathrm{tr}\{(-1)^F q^{L_0}\exp(2\pi\mathrm{i}\delta G_0)\exp(2\pi\mathrm{i}\beta V_0)\phi_0\ldots\phi_n\}$$

with the Grassmann-odd parameter β. Then we have

$$\langle V_0\phi_1\ldots\phi_n\rangle = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial\beta}\mathrm{tr}\{(-1)^F q^{L_0}\exp(2\pi\mathrm{i}\delta G_0)\exp(2\pi\mathrm{i}\beta V_0)\phi_0\ldots\phi_n\}\,.$$

The Ward identities involving the character-valued correlation functions can in principle be obtained by using (17) and the super W_3 algebra and these Ward identities are complete. However, the calculation will be very tedious and complicated and to get the compact expressions is, if not impossible, very difficult due to the non-linearity of the super W_3 algebra. Another way to determine the action of zero-mode J_0 inside the correlation functions is to use the coset construction for the super W_3 algebra [8] and the detailed study using this method will be given elsewhere.

Although the operator formalism we used is simple for deriving Ward identities on a supertorus in extended superconformal field theories, generalization of the formalism to general super Riemann surfaces is probably not straightforward and a possible way to do this is using the KN approach [34]. Deriving Ward identities for extended superconformal field theories on higher genus super Riemann surfaces would be an important problem.

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Appendix

In this appendix we give the super W_3 Ward identities for even spin structure s which are not given in the text. Here we define the correlation function by dividing by Z since it is not nilpotent in this case and we would like to compare results with those in the $N=0$ theories.

The Ward identity for one insertion of the generator $J(z,\theta)$ is

$$\begin{aligned}\langle J(z,\theta)\phi_1\dots\phi_N\rangle = \langle JX\rangle = \sum_{i=1}^{N}\Big\{ & \frac{v_i}{2}\partial_i^2 F_1(z,z_i,\theta,\theta_i|\tau) + W_{i1}\partial_i F_1(z,z_i,\theta,\theta_i|\tau)\partial_i \\ & + (W_{i1}' - W_{i1})D_i F_{01}\partial_i D_i + \tilde{W}_{i2}\partial_i^2 + W_{i3}F_1\Big[\sum_{j\neq i}(F_I(z_i,z_j|\tau)\partial_j \\ & + \tfrac{1}{2}D_j F_I D_j + \Delta_j\partial_j F_I) + z_i^{-2}\{-2z_i\partial_i + \tfrac{1}{2}\theta_i(-1+2\hat{\eta}_1)D_i \\ & + \Delta_i(-1+2\hat{\eta}_1)\}\Big]\Big\}\langle X\rangle + \theta\langle W_0(z)X\rangle \\ & + \sum_{i=1}^{N} W_{i3}F_1 z_i^{-2}\frac{\partial}{Z\partial(2\pi i\tau)}(Z\langle X\rangle) \end{aligned} \qquad \text{(A1)}$$

where

$$F_1(z,z_i,\theta,\theta_i|\tau) = \theta_i F_{H1}(z,z_i|\tau) + \theta F_{I1}(z,z_i|\tau)$$

$$F_{H1}(z,z_i|\tau) = z^{-3}z_i^2 f_0(z,z_i|\tau) \qquad F_{I1} = z^{-3}z_i^2 f(z,z_i|\tau)$$

$$F_{01} = \theta_i F_{H1} - \theta F_{I1}$$

$$f(e(z),e(z_i)|\tau) = \zeta(z,z_i|\tau) + 2\eta_1(\tau)z_i \qquad e(z) = \exp(2\pi i z)$$

$$f_0(z,z_i|\tau) = \frac{z_i}{z-z_i} + \left(\frac{z_i}{z}\right)^{1/2}\sum_{1/2,3/2\dots}\left[\frac{q^n}{1-q^n}\left(\frac{z_i}{z}\right)^n - \frac{q^n}{1-q^n}\left(\frac{z_i}{z}\right)^{-n}\right].$$

The Ward identities for two insertions of the generators are given by

$$\langle I(z,\theta)J(\omega,\varphi)\phi_1\ldots\phi_N\rangle = \langle IJX\rangle = \sum_{i=1}^{N}\Big\{F(z,z_i,\theta,\theta_i|\tau)\partial_i + \tfrac{1}{2}D_iF_iD_i$$
$$+ \Delta_i\partial_i F\Big\}\langle J(\omega,\varphi)X\rangle + \big[\tfrac{5}{2}\partial_\omega F(z,\omega,\theta,\varphi|\tau)$$
$$+ \tfrac{1}{2}D_{\omega\varphi}F_0D_{\omega\varphi}F(z,\omega,\theta,\varphi|\tau)\partial_\omega\big]\langle J(\omega,\varphi)X\rangle$$
$$+ \theta z^{-2}\frac{\partial}{Z\partial(2\pi i\tau)}(Z\langle J(\omega,\varphi)X\rangle) \tag{A2}$$

where

$$F(z,z_i,\theta,\theta_i|\tau) = F_I(z,z_i|\tau)\theta + \theta_iF_H(z,z_i|\tau)$$
$$F_0 = \theta_iF_H - \theta F_I \qquad F_I(z,z_i|\tau) = z^{-2}z_if(z,z_i|\tau)$$

and

$$F_H(z,z_i|\tau) = z^{-2}z_if_0(z,z_i|\tau) \tag{A3}$$
$$\langle J(z,\theta)J(\omega,\varphi)\phi_1\ldots\phi_N\rangle = \langle JJX\rangle$$
$$= \sum_{i=1}^{N}\Big\{\frac{v_i}{2}\partial_i^2F_1(z,z_i,\theta,\theta_i|\tau) + W_{i1}\partial_iF_1\partial_i + (W'_{i1}-W_{i1})D_iF_{01}\partial_iD_i$$
$$+ \tilde{W}_{i2}F_1\partial_i^2 + W_{3i}F_1\Big[\sum_{j\neq i}^{N}(F_I(z_i,z_j|\tau)\partial_j + \tfrac{1}{2}D_jF_ID_j + \Delta_j\partial_jF_I)$$
$$+ z_i^{-2}[-2z_i\partial_i + \tfrac{1}{2}\theta_i(-1+2\hat{\eta}_1)D_i + \Delta_i(-1+2\hat{\eta}_1)]\Big]\Big\}\langle JX\rangle$$
$$+ \sum_{i=1}^{N}W_{3i}F_0(z,z_i,\theta,\theta_i|\tau)z_i^{-2}\frac{\partial}{Z\partial(2\pi i\tau)}(Z\langle J(\omega,\varphi)X\rangle)$$
$$+ \frac{2}{21}\frac{1}{4!}\partial_\omega^{(4)}D_{\omega\varphi}F_{01}(z,\omega,\theta,\varphi|\tau)\langle X\rangle$$
$$+ \frac{1}{3!}\partial_\omega^{(3)}F_{01}(z,\omega,\theta,\varphi|\tau)\langle I(\omega,\varphi)X\rangle + \frac{1}{3}\frac{1}{2!}\partial_\omega^2D_{\omega\varphi}F_{01}D_{\omega\varphi}\langle IX\rangle$$
$$+ \tfrac{1}{3}\partial_\omega^2F_{01}\partial_\omega\langle IX\rangle + \tfrac{1}{6}\partial_\omega D_{\omega\varphi}F_{01}\partial_\omega D_{\omega\varphi}\langle IX\rangle$$
$$+ \partial_\omega F_{01}\big(\tfrac{1}{4}\partial_\omega^2\langle IX\rangle + \tfrac{7}{17}\langle\Gamma_0X\rangle + \tfrac{337}{408}\varphi\langle\Lambda X\rangle\big)$$
$$+ D_{\omega\phi}F_{01}\big(\tfrac{1}{20}\partial_\omega^2D_{\omega\varphi}\langle IX\rangle - \tfrac{7}{17}D_{\omega\varphi}\langle\Gamma_0X\rangle + \tfrac{21}{136}\langle\Lambda\rangle$$
$$+ \tfrac{28}{51}\varphi\langle KX\rangle\big) + F_1(z,\omega,\theta,\varphi|\tau)\big(\tfrac{14}{51}\langle KX\rangle + \tfrac{28}{51}D\langle\Lambda X\rangle + \tfrac{1}{15}\partial^3\langle IX\rangle\big)$$
$$+ \theta\langle W_0(z)J(\omega,\varphi)X\rangle \tag{A4}$$

where

$$\langle\Lambda(\omega)X\rangle = \sum_{i=1}^{N}\Big(F_I(\omega,z_i|\tau)\partial_i + \frac{1}{2}D_iF_ID_i + \partial_iF_I\Big)A + \frac{2}{\omega^2}(-1+\hat{\eta}_1)A - \frac{2}{\omega}\partial_\omega A$$
$$+\omega^{-2}\frac{\partial}{Z\partial(2\pi i\tau)}(ZA) + \frac{c}{12}\omega^{-4}\Big[-2-16\hat{\eta}_1+\sum_{n-1}^{\infty}\frac{2n^3q^n}{1-q^n}\Big]\langle X\rangle - \frac{3}{10}\partial_\omega^2 A \quad \text{(A5)}$$

with

$$A = \sum_{i=1}^{N}\Big\{F_I(\omega,z_i|\tau)\partial_i + \frac{1}{2}D_iF_ID_i + \Delta_i\partial_iF_I\Big\}\langle X\rangle + \omega^{-2}\frac{\partial}{Z\partial(2\pi i\tau)}(Z\langle X\rangle) \quad \text{(A6)}$$

$$\langle\Omega(\omega)X\rangle = 2\Big\{\sum_{j=1}^{N}[F_I(\omega,z_j|\tau)\partial_j + \tfrac{1}{2}D_jF_ID_j + \Delta_j\partial_jF_I\Big\}B$$
$$+\Big[\frac{3}{2\omega^2}(-1+2\hat{\eta}_1) - \frac{2}{\omega}\partial_\omega\Big]B + \omega^{-2}\frac{\partial}{Z\partial(2\pi i\tau)}(ZB) - \frac{3}{4}\partial_\omega^2 B \quad \text{(A7)}$$

with

$$B = \sum_{j=1}^{N}\{\theta_jF_H(\omega,z_j|\tau)\partial_j + \tfrac{1}{2}D_jF_H\theta_jD_j + \Delta_j\partial_jF_H\theta_j\Big\}\langle X\rangle \quad \text{(A8)}$$

$$\langle H(\omega,\varphi)\phi_i,\ldots\phi_N\rangle = \langle HX\rangle = 4\Big\{\sum_{i=1}^{N}(\theta_iF_H(\omega,z_i|\tau)$$
$$-\frac{1}{2}D_i\theta_iF_HD_i + \Delta\theta_i\partial_iF_H)\partial_\omega B + \frac{c}{12\omega^2}(2+\tilde{\eta}_1)\langle X\rangle - \frac{1}{2\omega^2}(2+\tilde{\eta}_1)A$$
$$+\frac{1}{\omega}\partial_\omega A\Big\} + \frac{17}{5c+22}\langle\Lambda X\rangle - \frac{7}{10}\partial_\omega^2 A \quad \text{(A9)}$$

and

$$\langle K(\omega)\phi_1\ldots\phi_N\rangle = \langle KX\rangle = 4\Big\{\sum_{i=1}^{N}\Big(F_I(\omega,z_i|\tau)\partial_i + \frac{1}{2}D_iF_ID_i + \Delta_i\theta_i\partial_iF_I\Big)\partial_\omega B$$
$$+\frac{3}{2}(-28\omega^{-3}+6\omega^{-3}\hat{\eta}_1)B - \frac{2}{\omega}\partial_\omega^2 B + \omega^{-2}\frac{\partial}{Z\partial(2\pi i\tau)}(Z\partial_\omega B)\Big\} - \frac{11}{10}\partial_\omega^3 B \quad \text{(A10)}$$

and the superconformal Ward identities which have been given in [29] and [30]. It is easy to see that the above Ward identities reduce to those in W_3 field theories [31] when setting the Grassman variables equal to zero.

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